TECHNICAL NOTE

The Nakayama-Koyama approach to laminar forced convection heat transfer to power-law fluids

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Received December 1987 and accepted for publication February 1988

The approximate Nakayama-Koyama integral approach is compared with an exact similarity solution for a free stream of the form $U \propto x^{1/2}$. It is demonstrated that the approximate method fails for highly pseudoplastic fluids ($n < n_{cr}$), and the variation of n_{cr} with the free-stream exponent m is provided. Finally, the applicability of the high apparent Prandtl number asymptote is considered, with emphasis on the stagnation point flow. For $n = 1.6$ the heat transfer rates are more than 10 times higher than those predicted by Nakayama and Koyama.

Keywords: heat transfer; forced convection; boundary layers; non-Newtonian fluids

Introduction

An approximate computational procedure for laminar forced convection heat transfer was first proposed by Nakayama *et al.* for Newtonian boundary layer problems[1] and later extended to non-Newtonian power-law fluids.[2] If the free-stream velocity belongs to the Falkner-Skan family,

$$\frac{U(x)}{U_\infty} = a\left(\frac{x}{L}\right)^m \tag{1}$$

the general equations for momentum and thermal energy reduce to simple algebraic expressions for the local wall friction and high apparent Prandtl number heat transfer rate.[3]

The purpose of the present note is threefolded. First, the approximate Nakayama-Koyama expression for the local skin friction coefficient is compared with accurate similarity solutions[4] for the particular parameter value $m = 1/2$. Then it is demonstrated that the Nakayama-Koyama approach fails for highly pseudoplastic fluids. Finally, the applicability of the high apparent Prandtl number asymptote for the heat transfer rate is discussed.

Momentum boundary layer problem

To provide a rapid and accurate calculation procedure for non-Newtonian boundary layer problems, the classical von Karman integral momentum approach was extended to the corresponding non-Newtonian problems.[2] Only inelastic fluids which obey the so-called power-law model

$$\tau = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{2}$$

were considered. In most boundary layer analyses, the wall shear stress τ_w is an essential variable, from which the local skin friction coefficient can be defined as $c_f \equiv 2\tau_w/\rho U^2$. For free streams of the particular form of Equation 1, this important parameter can be evaluated from the relation

$$c_f \, \mathrm{Re}_x^{1/(n+1)} = \left(\frac{mC^2}{6n\Lambda}\right)^{n/(n+1)} \tag{3}$$

where

$$C \equiv \frac{\delta}{U}\frac{\partial u}{\partial y}\bigg|_w \tag{4a}$$

$$\Lambda \equiv -\frac{1}{6}\frac{\delta^2}{U}\frac{\partial^2 u}{\partial y^2}\bigg|_w \tag{4b}$$

are shape factors, m is the free-stream exponent, and $\mathrm{Re}_x \equiv \rho x^n U^{2-n}/K$ is the local Reynolds number.

Assuming that the streamwise velocity component in the viscous boundary layer can be approximated by the Pohlhausen polynomial of fourth degree, the general integral equation is reduced to the simple algebraic relation:[2,3]

$$m = \frac{\Lambda G}{\frac{1+n}{6n}C - \Lambda\left(3nG + \frac{(1+n)(6-\Lambda)}{20}\right)} \tag{5}$$

where the shape factors C and G become

$$C = 2 + \Lambda \tag{6a}$$

$$G = \frac{148 - 8\Lambda - 5\Lambda^2}{1260} \tag{6b}$$

Thus, for given values of the Falkner-Skan flow parameter m and the power-law index n, the shape factor Λ is obtained iteratively from Equation 5. Subsequently, the local skin friction coefficient is evaluated from Equation 3.

Thermal boundary layer problem

Of particular interest in many industrial applications is the heat transfer rate between the wall and the fluid. The local heat

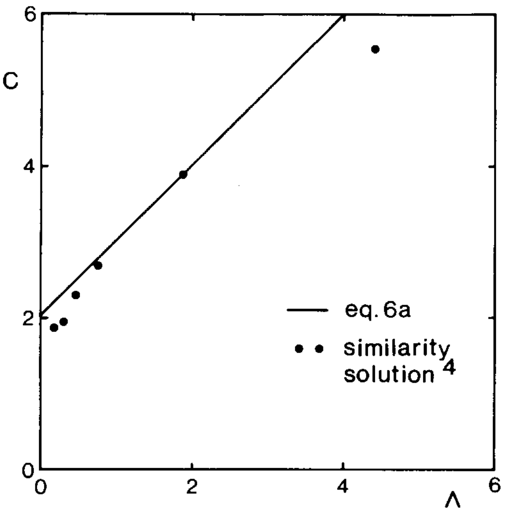

Figure 1 Shape factor C versus shape factor Λ for $m=1/2$

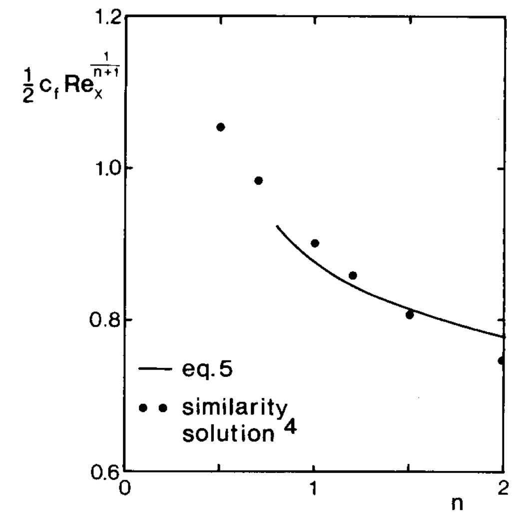

Figure 2 Variation of local skin friction coefficient with power-law index for $m=1/2$

transfer rate is then conveniently expressed as a local Nusselt number

$$\mathrm{Nu}_x \equiv -\frac{x}{T_w - T_0}\left.\frac{\partial T}{\partial y}\right|_w \tag{7}$$

where T_0 denotes the temperature in the bulk fluid. In accordance with the assumption for the velocity profile, the temperature profile in the thermal boundary layer was approximated by a fourth-degree polynomial.[2,3] Nakayama *et al.* were then able to derive simple asymptotic expressions for the local heat trasfer rate in the extreme Prandtl number cases. According to Nakayama and Koyama,[3] the Prandtl number is quite large for most practical problems involving non-Newtonian fluids, and the important high Prandtl number asymptote becomes

$$\mathrm{Nu}_x \mathrm{Re}_x^{-1/(n+1)} = \left[\left(\frac{m}{6nC^{n-1}\Lambda}\right)^{1/(n+1)} \frac{2C}{15} \cdot \left(1 + m - \frac{(1-n)(1-3m)}{3(1+n)}\right)\right]^{1/3} \mathrm{Pr}_x^{1/3} \tag{8}$$

where the local apparent Prandtl number is defined as

$$\mathrm{Pr}_x \equiv \frac{\rho c_p}{k}\left(\frac{K}{\rho}\right)^{2/(n+1)}\left(\frac{x}{U^3}\right)^{(1-n)/(1+n)} \tag{9}$$

While the wall temperature T_w is allowed to vary in the streamwise direction in the Nakayama-Koyama approach, only the case of an isothermal wall is considered here.

The heat transfer rate can now be evaluated from the explicit formulae (8) for given values of the flow parameter m and the power-law index n, once the shape factor Λ has been obtained from Equation 5.

Results and discussions

The momentum boundary layer equations for power-law fluids possess similarity solutions for free streams of the Falkner-Skan type (1).[5] An accurate finite-difference solution of the similarity problem for the particular parameter value $m=1/2$ has recently been provided by Andersson and Irgens.[4] The shape factors C and Λ derived from that solution are compared with the approximate result (6a) in Figure 1. The separation between the line and the symbols indicates how accurately the Pohlhausen profile approximates the exact similarity profile in the near-wall region. The resulting variation of the local skin friction coefficient with the power-law index is shown in Figure 2. The approximate results evaluated from Equation 5 closely approximate the

Notation

a	Free-stream parameter
c_f	Local skin friction coefficient
c_p	Specific heat
C	Shape factor
G	Shape factor
k	Thermal conductivity
K	Coefficient of consistency
L	Length scale; radius of cylinder
m	Free-stream parameter
n	Power-law index
Nu	Nusselt number
Pr	Apparent Prandtl number
Re	Reynolds number
T	Temperature
u	Streamwise velocity component in boundary layer
U	Free-stream velocity
x	Streamwise coordinate
y	Cross-stream coordinate

Greek symbols

δ	Momentum boundary layer thickness
Λ	Shape factor
ρ	Density
τ	Shear stress

Subscripts

cr	Critical value
0	Bulk fluid
w	Wall condition
x	Local value
∞	Approaching flow

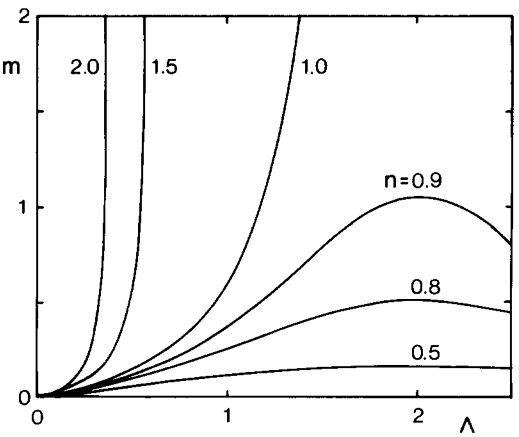

Figure 3 Variation of m with Λ evaluated explicitly from Equation 5 for different n values

similarity solutions over the n-range considered. The approximate results are slightly below the similarity solutions for pseudoplastics and somewhat above the similarity results for highly dilatant substances ($n \gtrsim 1.5$).

It may be observed, however, that approximate results are shown in Figure 2 only for dilatant fluids ($n > 1$) and slightly pseudoplastic fluids ($n \lesssim 1$). This is because the implicit expression 5 for Λ does not yield realistic solutions for arbitrary combinations of m and n. Figure 3 shows the variation of m with Λ for some different n values, as calculated explicitly from Equation 5. For the pseudoplastic fluids m is a monotonically increasing function of Λ in the range $0 < \Lambda < 2$. At $\Lambda = 2$, m reaches its maximum value and then starts to decrease as Λ is further increased.

For a given value of m the solution for Λ should be obtained as the intersection between the curve corresponding to the actual n value and the horizontal line m=constant. According to Figure 3, solutions can be obtained for any dilatant fluid, but only for slightly pseudoplastic substances. More specifically, solutions of Equation 5 are prohibited if the power-law exponent is below a certain critical limit n_{cr}. The variation of n_{cr} with m is displayed in Figure 4, which shows that n_{cr} is a monotonically increasing function of the Falkner-Skan flow parameter m. In the particular case $m = 1/2$ considered in Figures 1 and 2, for example, n_{cr} is slightly below 0.8.

The failure of Equation 5 to give solutions for Λ if $n < n_{cr}$ has not been observed by Nakayama and Koyama.[2,3] However, it is well known[6] that the Pohlhausen profile gives $u > U$ within the momentum boundary layer for $\Lambda > 2$, which is physically incorrect for steady flows. The shape factor Λ should therefore be restricted to values below 2, which incidentally correspond to the range $n \geqslant n_{cr}$.

The local apparent Prandtl number defined in Equation 9 plays an important role in the heat transfer analysis. For free streams of the form of Equation 1, the streamwise variation of Pr_x becomes[2,3]

$$Pr_x \propto x^{(1-3m)(1-n)/(1+n)} \tag{10}$$

which implies that $Pr_x \to 0$ as $x \to 0$ if $m < 1/3$ for pseudoplastics and if $m > 1/3$ for dilatant fluids. On the other hand, Pr_x tends to infinity as $x \to 0$ if $m > 1/3$ for the pseudoplastics and if $m < 1/3$ for the dilatant substances. The high apparent Prandtl number asymptote (8) should therefore be used near the stagnation point in the latter cases only.

In their recent paper,[3] Nakayama and Koyama applied the high apparent Prandtl number asymptote to estimate the heat flux in the stagnation region on a circular cylinder in crossflow. The experimental investigation by Shah *et al.*[7] on a cylinder immersed in a non-Newtonian crossflow, showed that the free-stream velocity can be approximated as

$$\frac{U(x)}{U_\infty} = 2\left[0.92\left(\frac{x}{L}\right) - 0.131\left(\frac{x}{L}\right)^3\right] \tag{11}$$

where U_∞ is the uniform stream velocity approaching the cylinder and L is the cylinder radius. Near the stagnation point $x = 0$ the second term can be neglected and Equation 11 reduces to the Falkner-Skan form (1) with $m = 1$ and $a = 1.84$, i.e., stagnation point flow. Nakayama and Koyama[3] evaluated the heat transfer from the cylinder to a dilatant fluid ($n = 1.6$) from Equation 8. However, it is evident from Equation 10 that Pr_x tends to zero rather than to infinity at the stagnation point, and the high apparent Prandtl number asymptote (8) should not be used in the stagnation point area for the dilatant fluids. The relevant expression for the local heat transfer rate is therefore the low apparent Prandtl number asymptote

$$Nu_x Re_x^{-1/(n+1)} = \left(\frac{3(m+1)}{10}\right)^{1/2} Pr_x^{1/2} \tag{12}$$

derived by Nakayama *et al.*[2] This result can be rewritten as

$$Nu\, Re^{-1/(n+1)} = \left(\frac{3a(m+1)Pr}{10}\right)^{1/2} \left(\frac{x}{L}\right)^{(m-1)/2} \tag{13}$$

where

$$Nu \equiv -\frac{L}{T_w - T_0} \frac{\partial T}{\partial y}\bigg|_w = \frac{L}{x} Nu_x \tag{14a}$$

$$Re \equiv \frac{\rho L^n U_\infty^{2-n}}{K} \tag{14b}$$

$$Pr \equiv \frac{\rho c_p}{k} \left(\frac{K}{\rho}\right)^{2/(n+1)} \left(\frac{L}{U_\infty^3}\right)^{(1-n)/(1+n)} \tag{14c}$$

denote an alternative local Nusselt number, and characteristic Reynolds and Prandtl numbers, respectively.

In order to make comparisons with the predictions of Nakayama and Koyama,[3] the local heat transfer rate is evaluated from the low apparent Prandtl number estimate (13) with $m = 1$ and $a = 0.92$. (Unfortunately, the factor 2 in the empirical free-stream distribution (11) has been overlooked by some investigators.[3,8]) For the particular case of stagnation point flow ($m = 1$) the heat transfer grouping $Nu\, Re^{-1/(n+1)}$ becomes independent of x. For $Pr = 10$ and 100, for instance,

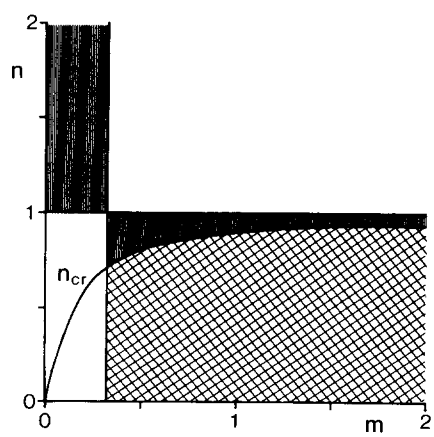

Figure 4 Range of applicability. Shaded areas indicate the range over which the Nakayama-Koyama high Prandtl number expressions are directly applicable. The asymptote (8) can also be applied in the crosshatched range.

$\mathrm{Nu}\,\mathrm{Re}^{-1/(n+1)}$ is equal to 2.349 and 7.430, respectively. These heat transfer rates are more than 10 times higher than those obtained by Nakayama and Koyama using the high apparent Prandtl number asymptote (8).

The low apparent Prandtl number asymptotes, Equations 12 and 13, which apply in the stagnation region for dilatant substances, are independent of the power-law index n. This is because the ratio of the momentum boundary layer thickness to the thermal boundary layer thickness tends to zero as $\mathrm{Pr}_x \to 0$. Physically, this means that the temperature field adjusts from T_w to T_0 in fluid with velocity $U(x)$. Thus, as a first approximation, the viscous boundary layer does not contribute to the heat flux through the wall, and the local Nusselt number becomes independent of n according to Equations 12 and 13. More accurate approximations for Newtonian fluids, taking into account the displacing effect of the momentum boundary layer, have been discussed by Andersson.[9] It can thus be anticipated that an exact solution for the heat transfer grouping $\mathrm{Nu}\,\mathrm{Re}^{-1/(n+1)}$ will depend on n except at the stagnation point $x=0$.

Finally, we mention that exact similarity solutions of the heat transfer problem are possible only for $m=1/3$, while similarity solutions of the momentum boundary layer problem can be obtained for any $m \geqslant 0$. Improved accuracy of the heat transfer estimate (8) can therefore be achieved if the shape factors Λ and C defined in Equation 4 are taken from accurate similarity solutions of the momentum boundary layer problem, e.g., those provided by Hsu,[10] rather than from the approximate relation (5). Moreover, the failure of Equation 5 for $n<n_{cr}$ is thereby circumvented.

Conclusions

It has been demonstrated that the Nakayama-Koyama approach accurately predicts the local skin friction coefficient for the free stream $U \propto x^{1/2}$. Moreover, it is revealed that no solution can be obtained for $n<n_{cr}$. Full application of the Nakayama-Koyama expressions for high apparent Prandtl numbers is therefore possible only for certain combinations of the power-law index n and the free-stream exponent m, as indicated by the shaded areas in Figure 4. However, if the shape factors C and Λ can be obtained by any other means, the high apparent Prandtl number asymptote (8) is applicable also in the cross-hatched area in Figure 4.

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